COMPOSITION OPERATORS IN THE LIPSCHITZ SPACE OF THE POLYDISCS

ZHONGSHAN FANG AND ZEHUA ZHOU*

Abstract. In 1987, Shapiro shew that composition operator induced by symbol φ is compact on the Lipschltz space if and only if the infinity norm of φ is less than 1 by a spectral-theoretic argument, where φ is a holomorphic self-map of the unit disk. In this paper, we shall generalize Shapiro's result to the n-dimensional case.

1. Introduction

Let U^n be the unit polydiscs of n-dimensional complex spaces C^n with boundary ∂U^n, the class of all holomorphic functions on domain U^n will be denoted by $H(U^n)$. Let $\varphi(z) = (\varphi_1(z), \cdots, \varphi_n(z))$ be a holomorphic self-map of U^n, composition operator is defined by

$$C_\varphi(f)(z) = f(\varphi(z))$$

for any $f \in H(U^n)$ and $z \in U^n$.

In the past few years, boundedness and compactness of composition operators between several spaces of holomorphic functions have been studied by many authors: by Jarchow and Ried [4] between generalized Bloch-type spaces and Hardy spaces, between Bloch spaces and Besov spaces and BMOA and VMOA in Tian's thesis [10].

More recently, there have been many papers focused on studying the same problems for n-dimensional case : by Zhou and Shi[15][16][17] on the Bloch space in polydisk or classical symmetric domains, Gorkin and MacCluer [3] between hardy spaces in the unit ball.

For the Lipschitz case, the compactness of C_φ is characterized by "little-oh" version of Madigan's [6] the bounedness condition, the same results in polydisc were obtained by Zhou [11] and by Zhou and Liu

2000 *Mathematics Subject Classification.* Primary: 47B38; Secondary: 26A16, 32A16, 32A26, 32A30, 32A37, 32A38, 32H02, 47B33.

Key words and phrases. Composition operator, Lipschitz space, Polydiscs, Several complex variables.

*Zehua Zhou, corresponding author. Supported in part by the National Natural Science Foundation of China (Grand Nos. 10671141, 10371091).

[14]. In all these works the main goal is to relate function theoretic properties of ϕ to boundedness and compactness of C_ϕ.

To our surprise, by a spectral-theoretic argument, Shapiro [9] obtained the following fact: C_φ is compact on the Lipschltz space $L_\alpha(D)$ if and only if $||\varphi||_\infty < 1$. In this paper, we shall generalize Shapiro's result to the unit polydisc.

2. Notation and background

Throughout the paper, D is the unit disk in one dimensional complex plane, and $|||z||| = \max_{1\le j\le n}\{|z_j|\}$ stands for the sup norm on the unit polydisc. Define $Rf(z) =< \nabla f(z), \bar{z} >$ where $z = (z_1, \cdots, z_n) \in U^n$, and $H(U^n, D)$ for the class of the holomorphic mappings from U^n to D. For $0 < \alpha < 1$, it is well known that the Lipschitz space $L_{1-\alpha}(U^n)$ is equivalent to $\alpha - Bloch$ space, which is defined to be the space of holomorphic functions $f \in U^n$ such that

$$||f||_{1-\alpha} = \sup_{z\in U^n} \sum_{j=1}^{n} (1 - |z_j|^2)^\alpha |\frac{\partial f}{\partial z_j}(z)| < \infty.$$

Here, Lipschitz space $L_{1-\alpha}(U^n)$ is a Banach space with the equivalent norm

$$||f|| = |f(0)| + ||f||_{1-\alpha}.$$

The Kobayashi distance k_{U^n} of U^n is given by

$$k_{U^n}(z, w) = \frac{1}{2}\log\frac{1 + |||\phi_z(w)|||}{1 - |||\phi_z(w)|||},$$

where $\phi_z : U^n \to U^n$ is the automorphism of U^n given by

$$\phi_z(w) = (\frac{w_1 - z_1}{1 - \overline{z_1}w_1}, \cdots, \frac{w_n - z_n}{1 - \overline{z_n}w_n})$$

Since the map $t \to \log\frac{1+t}{1-t}$ is strictly increasing on $[0, 1)$, it follows that

$$k_{U^n}(z, w) = \max_{1\le j\le n}\{\frac{1}{2}\log\frac{1 + |\frac{w_j - z_j}{1-\overline{z_j}w_j}|}{1 - |\frac{w_j - z_j}{1-\overline{z_j}w_j}|}\} = \max_{1\le j\le n}\{\rho(z_j, w_j)\},$$

where ρ is the Poincaré distance on the unit disk $D \subset C$.

Following [1], the horosphere $E(x, R)$ of center $x \in \partial U^n$ and radius R and the Korányi region $H(x, M)$ of vertex x and amplitude M are defined by

$$E(x, R) = \{z \in U^n : \limsup_{w\to x}[k_{U^n}(z, w) - k_{U^n}(0, w)] < \frac{1}{2}\log R\}$$

and

$H(x,M)=\{z\in U^n : \limsup_{w\to x}[k_{U^n}(z,w)-k_{U^n}(0,w)]+k_{U^n}(0,z)<\log M\}$.

We say that f has $K-limit$ $L\in C$ at x if $f(z)\to L$ as $z\to x$ inside any Korányi region $H(x,M)$, we shall write $\widetilde{K}-\lim_{z\to x} f(z)=L$.

Let $f\in H(U^n,D)$ and $x\in\partial U^n$. If there is δ such that

$$\liminf_{w\to x}\frac{1-|f(w)|}{1-|||w|||}=\delta<\infty,$$

we call f is $\delta-Julia$ at x. If there exists $\tau\in\partial U^n$ such that

$$f(E(x,R))\subseteq E(\tau,\delta R)$$

for all R, we call this τ is the restricted E-limit of f at x.

It should be noticed that $\delta>0$. In fact,

$$\rho(0,f(w))\le\rho(0,f(0))+\rho(f(0),f(w))\le\rho(0,f(0))+k_{U^n}(0,w);$$

therefore $\frac{1-|f(w)|}{1-|||w|||}\ge\frac{1-|f(0)|}{2(1+|f(0)|)}>0$.

3. Some Lemmas

Lemma 1. *(Julia-Wolff-Carathéodory Theorem, Theorem 4.1 in* [1]*) Let $f\in H(U^n,D)$ be $\delta-Julia$ at $x\in\partial U^n$, and $\tau\in\partial U$ be the restricted E-limit of f at x, then*

$$\widetilde{K}-\lim_{z\to x}\frac{\partial f}{\partial x}(z)=\delta\tau.$$

Lemma 2. *(Theorem 1 in* [11] *or Corollary 4.1 in* [14]*) Composition operator C_φ is bounded on the Lipschitz space $L_{1-\alpha}(U^n)$ if and only if there is a constant $M>0$ such that*

$$\sum_{k,l=1}^{n}\left|\frac{\partial\phi_l}{\partial z_k}(z)\right|\left(\frac{1-|z_k|^2}{1-|\phi_l(z)|^2}\right)^\alpha\le M$$

for $z\in U^n$.

Lemma 3. *(Theorem 2 in* [11] *or Corollary 4.2 in* [14]*) Composition operator C_φ is compact on the Lipschitz space $L_{1-\alpha}(U^n)$ if and only if*

$$\lim_{\delta\to 0}\sup_{dist(\varphi(z),\partial U^n)<\delta}\sum_{k,l=1}^{n}|\frac{\partial\varphi_l}{\partial z_k}(z)|\frac{(1-|z_k|^2)^\alpha}{(1-|\varphi_l(z)|^2)^\alpha}=0.$$

Lemma 4. *(Lemma 3.2 in* [1]*) Let $f\in H(U^n,D)$ and $x\in\partial U^n$. Then*

$$\liminf_{w\to x}\frac{1-|f(w)|}{1-|||w|||}=\liminf_{t\to 1^-}\frac{1-|f(\varphi_x(t))|}{1-t},$$

where $\varphi_x(z)=zx$ for any $z\in D$.

4. Main theorem

Theorem 1. *Suppose C_φ is bounded on $L_{1-\alpha}(U^n)$, then for every $1 \leq l \leq n$ and $\xi \in \partial U^n$ with $|\varphi_l(\xi)| = 1$, φ_l is $\delta - Julia$ at ξ.*

Proof. For every $1 \leq l \leq n$ and $\xi \in \partial U^n$ with $\varphi_l(\xi)| = \eta$ and $\eta = e^{\theta_0}$, we will show that φ_l is $\delta - Julia$ at ξ according to the following cases.

Case 1: $\xi = (\xi_1, \xi'), \xi_1 = e^{\theta_1}$ and $|||\xi'||| < 1$.

First we consider the special case for $\xi = e_1 = (1, 0, \cdots, 0)$ and $\eta = 1$.

For $r \in (1/2, 1)$, define $\sigma(r) = (r, 0, \cdots, 0) = re_1$ such that

$$\lim_{r \to 1^-} \varphi_l(\sigma(r)) = 1.$$

Setting $g(r) = \varphi_l(re_1)$, then $g'(r) = \frac{\partial \varphi_l}{\partial z_1}(re_1)$. It follows from Lemma 2 that the boundedness of C_φ implies that

$$h(r) = R\varphi_l(re_1)(\frac{1-r}{1-\varphi_l(re_1)})^\alpha = rg'(r)(\frac{1-r}{1-g(r)})^\alpha$$

is bounded.

Putting $u(r) = \frac{1-g(r)}{1-r}$, it is easy to see that $g'(r) = -(1-r)u'(r)+u(r)$ and

$$h(r) = ru(r)^{-\alpha}[-(1-r)u'(r) + u(r)].$$

If we write $v(r) = u(r)^{1-\alpha}$, then

$$-\frac{1}{1-\alpha}(1-r)v'(r) + v(r) = \frac{h(r)}{r}$$

the general solution of this differential equation is

$$v(r) = -\frac{1-\alpha}{(1-r)^{1-\alpha}} \int_1^r \frac{h(s)}{s(1-s)^\alpha} ds + \frac{C}{(1-r)^{1-\alpha}}.$$

Since h is bounded, the first term in the right above is a bounded function of r, and moreover $v(r)$ is of the order $o(\frac{1}{(1-r)^{1-\alpha}})$ as $r \to 1^-$, so we have $C = 0$. Hence v, and moreover u is also bounded, according to Lemma 4, for some δ, φ_l is $\delta - Julia$ at e_1.

Now we return to the proof in case 1. Considering the mapping $\tilde{\varphi}_l : U^n \to U^n$, where

$$\tilde{\varphi}_l(z_1, z') = e^{-i\theta_0} \cdot \varphi_l(e^{i\theta_1} z_1, \phi_{\xi'}(z'))$$

for $z = (z_1, z') \in U^n$. It is easy to check that $C_{\tilde{\varphi}_l}$ is bounded on $L_{1-\alpha}(U^n)$ and $\tilde{\varphi}_l(e_1) = 1$.

By the above argument, we get $\liminf_{t\to 1^-} \frac{1-|\tilde{\varphi}_l(te_1)|}{1-t} = \delta < +\infty$, that is

$$\begin{aligned}\liminf_{t\to 1^-} \frac{1-|\varphi_l(t\xi_1, \xi'))|}{1-t} &= \liminf_{t\to 1^-} \lim_{r\to 1^-} \frac{1-|\varphi_l(t\xi_1, r\xi'))|}{1-t} \\ &\geq \liminf_{t\to 1^-} \frac{1-|\varphi_l(t\xi_1, t\xi'))|}{1-t}.\end{aligned}$$

It follows from Lemma 4 that

$$\liminf_{w\to\xi} \frac{1-|\varphi_l(\xi)|}{1-|||\xi|||} = \delta < +\infty.$$

Case2: $\xi = (\xi_1, \xi_2, \xi'), \xi_1 = e^{\theta_1}, \xi_2 = e^{\theta_2}$ and $|||\xi'||| < 1$.

Now assume $\varphi_l(1, 1, 0, \cdots, 0) = 1$, and set $g(r) = \varphi_l(r, r, 0, \cdots, 0)$ for $r \in (1/2, 1)$. then $g'(r) = \frac{\partial\varphi_l}{\partial z_1}(r, r, 0, \cdots, 0) + \frac{\partial\varphi_l}{\partial z_2}(r, r, 0, \cdots, 0)$, and so $R\varphi_l(r, r, 0, \cdots, 0) = rg'(r)$, we can deal with it as in the case 1, and we can get u is bounded, furthermore

$$\liminf_{w\to\xi} \frac{1-|\varphi_l(\xi)|}{1-|||\xi|||} = \delta < +\infty.$$

Case 3: For the case $\varphi_l(\xi) = 1$ with $\xi = \sum_{k=1}^{n} \beta_k e_k$, where $\beta_k = 0$ or 1, and $e_k = (0, , 0, \cdots, 1, 0, \cdots, 0)$ with the $k-th$ component is 1, otherwise 0; and even more general case, in a similar argument with the cases 1 and 2, we can also show

$$\liminf_{w\to\xi} \frac{1-|\varphi_l(\xi)|}{1-|||\xi|||} = \delta < +\infty.$$

This completes the proof of this theorem. □

Theorem 2. *C_φ is compact on $L_{1-\alpha}(U^n)$ if and only if $\varphi_j \in L_{1-\alpha}(U^n)$ and $||\varphi_j||_\infty < 1$ for each $j = 1, 2, \cdots, n$.*

Proof. Sufficiency is obvious. Now we just turn to the necessity. Suppose to the contrary that there exists l $(1 \leq l \leq n)$ satisfying $|\varphi_l(\xi)| = 1$ for some $\xi \in \partial U^n$. It follows from Theorem 1 that φ_l is $\delta - Julia$ at ξ, therefore by Lemma 1, we have $R\varphi_l(z)$ has $K - limit$ at ξ. Hence

$$\sum_{k,l=1}^{n} |\frac{\partial\varphi_l}{\partial z_k}(z)| \frac{(1-|z_k|^2)^\alpha}{(1-|\varphi_l(z)|^2)^\alpha}$$

$$
\begin{aligned}
&\geq \sum_{k,l=1}^{n} |\frac{\partial \varphi_l}{\partial z_k}(z)| \frac{(1-|||z|||^2)^{\alpha}}{(1-|\varphi_l(z)|^2)^{\alpha}} \\
&\geq \sum_{k,l=1}^{n} |z_k \cdot \frac{\partial \varphi_l}{\partial z_k}(z)| \frac{(1-|||z|||^2)^{\alpha}}{(1-|\varphi_l(z)|^2)^{\alpha}} \\
&\geq C \sum_{l=1}^{n} |R\varphi_l(z)| \frac{(1-|||z|||)^{\alpha}}{(1-|\varphi_l(z)|)^{\alpha}} \\
&\geq C\delta^{1-\alpha}
\end{aligned}
$$

as $z \to \xi$ inside any Korányi region, where we can take $C = \frac{1}{2^\alpha}$. It is a contradiction to the compactness of C_φ by Lemma 3. Now the proof of Theorem 2 is completed. □

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Department of Mathematics
Tianjin Polytechnic University
Tianjin 300160
P.R. China.
E-mail address: `fangzhongshan@yahoo.com.cn`

Department of Mathematics
Tianjin University
Tianjin 300072
P.R. China.
E-mail address: `zehuazhou2003@yahoo.com.cn`